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Tuning: At the Crossroads

Introduction

The arena of musical scales and tuning has certainly not been a quiet place to be for the past three hundred years. But it might just as well have been if we judge by the results: the same $12\sqrt{2}$ equally tempered scale established then as the best available tuning compromise, by J. S. Bach and many others (Helmholtz 1954; Apel 1972), remains to this day essentially the *only* scale heard in Western music. That monopoly crosses all musical styles, from the most contemporary of jazz and avant-garde classical, and musical masterpieces from the past, to the latest technopop rock with fancy synthesizers, and everywhere in between. Instruments of the symphony orchestra attempt with varying degrees of success to live up to the 100-cent semitone, even though many would find it inherently far easier to do otherwise: the strings to "lapse" into Pythagorean tuning, the brass into several keys of Just intonation (Barbour 1953). And these easily might do so were it not for the constant viligance on the part of performers, and the readily available yardsticks for equal temperament provided by the woodwinds to some extent, but more so by the harp, organ, or omnipresent piano (inexact standards that they may in truth be).

Yet this apparent lack of adventurousness is not due to any lack of good alternatives (Olson 1967; Backus 1977; Lloyd and Boyle 1979; Bateman 1980; Balzano 1980) or their champions. Indeed an experienced musician would have to be preposterously naïve, sheltered, and deaf (!) not to have encountered at least a name or two like Yasser (1975) or Partch (1979), or in an earlier era, Bosanquet, White, Brown, or General Thompson (Helmholtz 1954; Partch 1979). These pioneers were certainly not known for their shy reticence on behalf of their various tuning reform proposals. Nearly all built or planned the construction of instruments that performed within the new "tuning of choice," and all published papers or books demonstrating the superiority of their new scales in at least some way over equal temperament. The tradition has continued with Yunik and Swift (1980), Blackwood (1982), and the present author (Milano 1986), and shows no sign of slowing down despite the apparent apathy with which the musical mainstream has regularly greeted each new proposal.

Of course there's a perfectly reasonable explanation for the mainstream's evident preference to remain "rut-bound" when by now there are at least a dozen clearly better-sounding ways to tune our scales, if only for at least part of the time: it requires a lot of effort of several kinds. I'm typing this manuscript using a Dvorak keyboard (for the first time!), and I assure you it's not easy to unlearn the QWERTY habits of a lifetime, even though I can already feel the actual superiority of this unloved but demonstrably better keyboard. It's been around since 1934, and only now with computers and speed and accurate data-entry tasks such as "directory assistance" is there the slightest chance it may be resurrected as a long overdue replacement for the deliberately slow (no kidding!) layout developed by Sholes over a century ago.

Musical keyboards are a fundamental part of the West's music-making culture, so keyboards need to be addressed with regard to compatibility with any existent or forthcoming tuning schema. Clearly the fretless string family is almost unrestricted as to the particular scales and tuning that can be used (the musician's ears and training are rather a different matter). This is sufficient reason for the historical concern given to new and modified keyboard designs by past proponents of tuning reform (Bosanquet, White, Yasser, etc.), although in practice Partch and others have frequently "made do" with the standard seven-white, five-black.

But as with Dvorak replacing QWERTY, it's difficult to challenge any sort of standard, once that standard has persisted for more than one generation. We all tend to forget the precariousness with which all standards are birthed, and grant those that come before us a sacrosanct status which is likely unjustifiable, and which the original designers might, if alive today, find quite laughable.

Tuning and the Zealot

No doubt frustrations do spawn a missionary zeal, and there is ample evidence that all of the pioneers we've been talking about had their fair share of frustration and zeal. It's both instructive and amusing to read, back-to-back, the books on tuning by Partch (1979) and Barbour (1953). Partch was arguably the greatest champion of Just intonation of this century, and his writing frequently lapses into often humorous sarcasm:

> Name-Your-Octave-Pay-Your-Pounds-and-Take-It-Away Jennings (Partch 1979, p. 394)

and emphatic impatience with those who seem most unwilling to accept the clear superiority of Just tuning:

> . . . and the ear does not budge for an instant from its demand for a modicum of consonance in harmonic music nor enjoy being bilked by near-consonances which it is told to hear as consonances. The ear accepts substitutes against its will (Partch 1979, p. 417).

While understandable, the pervasive undertone is eventually wearying (even if most often justified) and does little to attract complete sympathy from a neutral reader. Barbour's classic study (which Partch generously praises) is no less subtlely dogmatic, although in this case the a priori ideal is not Just intonation, but equal temperament. He calculates every alternative tuning's "deviation" from $12\sqrt{2}$, and argues most on behalf of those which exhibit the least mean and standard deviations from equal temperament. There's something uncomfortably circular in the reasoning throughout the book, sort of like:

Given, equal temperament
Therefore, equal temperament
Q. E. D.

This unfortunate quality of "sentence first, verdict afterwards" places Barbour's otherwise laudable achievement into the same zealot's pulpit with Partch, Yasser, Bosanquet, Poole, Brown, Perret, Captain Herschel, General Thompson, and others.

Barbour's *Tuning and Temperament* also suffers from several small errors:

> "Furthermore, Cx^{-3} and $E\sharp^{-3}$ differed by only six cents . . ." (p. 112). The second note ought be $E\sharp^{-2}$.
>
> ". . . ingenious mechanism by which D♭ and A♭ could be substituted for C♯ and D♯ . . ." (p. 108). But G♯ is the fifth above C♯, not D♯.
>
> ". . . and (the log of) $2^{18/31}$, .1757916100, . . ." (p. 119). But the correct log is $.17\underline{4}7916100$.

and, in light of many more as above, from hypocracy:

> "There were, as usual with Kircher, many errors . . ." (p. 110).

and from sarcasm:

> "Only in the design of the keyboards did the inventors show their ingenuity, an ingenuity that might better have been devoted to something more practical" (p. 113).

which quite ignores the ingenuity spent developing the traditional seven-white, five-black keyboard Barbour accepts as a "given," to say nothing of the merits of many possible good new scales that really do require effort and ingenuity to find. Talk about a closed mindset! And Barbour represents among musical theorists the best of the knowledgeable "champions" on behalf of equal temperament. This certainly seems to be a subject that attracts passionate words (see Lloyd and Boyle 1979, Chapter 8 for even more) from all sides!

The Crossroads

It may seem unfair to single out Barbour as above, although somehow I wince more from his and other theorists' polemic on behalf of "the haves" of status quo versus "the have nots" of anyone who might not totally agree with the final sentence of Barbour's book:

> Perhaps the philosophical Neidhardt should be allowed to have the last word on the subject: "Thus equal temperament carries with itself its comfort and discomfort, like the holy estate of matrimony."

That's a cute quote, but its use here is not a little smug, suggesting what was "good enough for Grandpa" ought be good enough in perpetuity for the rest of us. There's no thought I've read by even the most messianic of tuning reformists that raises my hackles as much.

The truth of the matter is that up until now there's simply been no way to investigate beyond the standard scale within the limits of the precision of the available technology. Acoustic instruments are almost compatible with a $\pm 5-10$-cent pitch tolerance at best. A fine grand piano cannot be expected to perform much better (Helmholtz 1954, pp. 485–493), not to overlook its particularly pronounced octave stretching (Backus 1977, p. 292; Benade 1976, pp. 313–322), a phenomenon we will examine later. Even the best analog synthesizers cannot do much better, and up until recently did a lot worse (I shudder to remember the constant pitch drifts while realizing *Switched-on Bach* on an early Moog!).

More frustrating than that is the usual limit on number of notes in an octave, which really was the wholly "economic" reason for musicians of Bach's time to yield the much preferable sounds of the several varieties of meantone tuning for $12\sqrt{2}$ equal temperament. Ellis describes the results in his fine appendixes to Helmholz's book (1954, p. 434):

> If carried out to 27 notes . . . it would probably have still remained in use. . . . The only objection . . . was that the organ-builders, with rare exceptions . . . used only 12 notes to the octave . . . and hence this temperament [Meantone] was first styled "unequal" (whereas the organ, not the *temperament* was—not unequal, but—*defective*) and then abandoned.

Not everyone agreed with abandoning meantone. The English held out on many instruments until the end of the last century. Even now the truce of equal temperament is not altogether as benign as all of us born into it commonly assume:

> The first of these pieces contained a number of sustained major thirds, which work perfectly well on an organ tuned to one of the unequal temperaments common in the seventeenth century, but which fight unmercifully on today's equally tempered instruments. During the playing of it the audience stirred uneasily, and, when I have played the tape, numerous musicians . . . have asked me what terrible thing went wrong with the organ. Most are incredulous when the explanation is given, even when they listen to the piece by Bach played the same evening on the identical organ, sounding . . . like the admirable instrument it is, Bach . . . arranged for his thirds to come and go, well disguised by their musical context." (Benade 1976, pp. 312–313).

Computer-controlled synthesis, on the other hand, has no inherent need to respect these heretofore inescapable limitations. There's certainly little need here to provide any further motivation for breaking out of the three-century "rut" that this long introductory material has documented. The exponential growth in computers has finally expanded to include systems designed expressly for music production, editing, and performance at low enough cost to be as affordable as, say, a good grand piano. This is the first time instrumentation exists that is both powerful enough and convenient enough to make practical the notion: any possible timbre, in any possible tuning, with any possible timing, sort of a "three T's of music." That places us at a crossroads, to figure out just how to use all of this newly available control. And we'll discover that the three "T's" are really tied together.

Musical Timbres

In Figs. 1–3 we have spectral plots for three common types of musical timbre. Figure 1 depicts purely harmonic partials (the most common, at least in the West), such as heard on the horn and most wind and string instruments. The equations of vibration here are usually quite elementary:

$$f_n = nf_1$$

Fig. 1. Original: pure harmonic partials.

Fig. 2. Original: stretched harmonic partials.

Fig. 3. Original: nonharmonic partials.

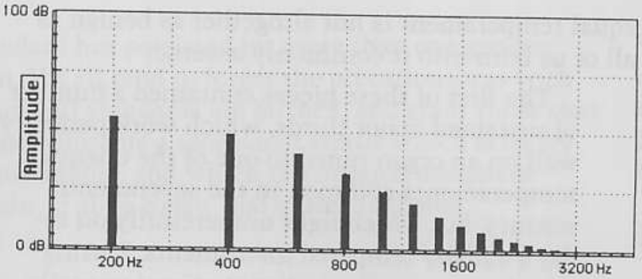
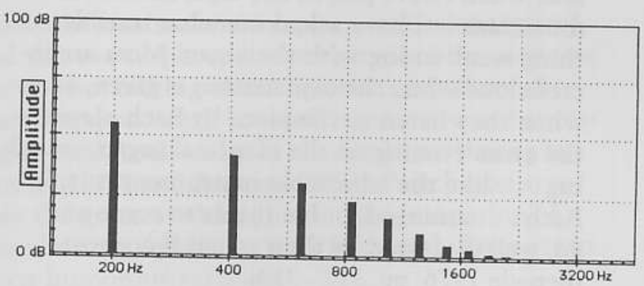


Fig. 2



which gives the frequency of the nth partial, f_n, as the simple integer n multiples of the "fundamental frequency," f_1. For a perfectly idealized vibrating string the expression is:

$$f_n = n(1/Lr)\sqrt{(T/d)}\sqrt{(1/4\pi)}$$

where the given (round) string has a length L, a radius r, a density d, and is under tension T (Benade 1976, p. 313). We can see at once that this is merely a more complete form of the first equation.

Figure 2 depicts the stretched harmonic partials heard on most "real world" thicker strings, as the lowest strings of the 'cello, but most famously on the piano. This explains the comment made earlier about the piano being a less-than-ideal pitch reference. The equations of motion of such instruments yield an overtone sequence described by:

$$f_n = nf_1 * (1 + Jn^2)$$

which is the same as the original equation, except that the Jn^2 term gradually raises the successive partial frequencies above the nf_1 "fundamental." J is a small coefficient given by:

$$J = (r^4 Y/TL^2)(\pi/2)^3$$

and on a good piano will have a value close to .00016 for "middle-C." J increases fairly uniformly by a factor of 2.76 for every octave we rise above middle-C, and similarly decreases by the same factor as we go down each octave (Benade 1976, p. 315).

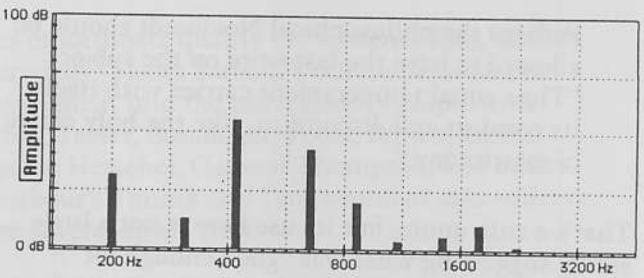


Figure 3 depicts the nonharmonic partials heard on more complex vibrating bodies, primarily the *ideophones* of the percussion family, although a fascinating world of new timbres is now possible by combining properties of "blown," "bowed," and "sustained" from strings and winds with the more complicated partial structure of ideophones. (The beginnings of this sort of work on *timbral hybrids* is discussed in part in [Milano 1986].) For a hinged bar the partials are described:

$$f_n = n^2(r/L^2)\sqrt{(Y/d)}(\pi/\sqrt{2}).$$

Here, as in the previous equation, Y is the *modulus of elasticity,* and L and d are the length and density, just as with a string.

More complex vibrating bodies have fairly involved mathematical descriptions which we will not go into here. For the interested reader an excellent treatment can be found in *The Theory of Sound* (Rayleigh 1945, pp. 255–432 of Vol. I).

In the case of a rectangular bar (Benade 1976, p. 51), n varies:

$$n_1 \approx 1.00$$
$$n_2 \approx 2.68$$
$$n_3 \approx 3.73$$
$$n_4 \approx 5.23$$

which, it will be noted, form a series of partials further apart than any of the previous timbres. In the case of a vibrating circular membrane, the sequence, here of a well-behaved timpani (Benade 1976, p. 144; also similarly Rayleigh 1945, p. 331), looks like:

$$
\begin{aligned}
n_1 &\approx 1.000\\
n_2 &\approx 1.504\\
n_3 &\approx 1.742\\
n_4 &\approx 2.000\\
n_5 &\approx 2.245\\
n_6 &\approx 2.494\\
n_7 &\approx 2.800
\end{aligned}
$$

which, it will be noted, form a series of partials closer together than any of the previous timbres. (Note how n_1, n_2, n_4, and n_6 look when each term is multiplied by two, implying that the timpani has a "missing" theoretical fundamental. This inspired me to try synthesizing a "basstimp" in which this implied partial was added, with a very agreeable result!)

A Confluence of Partials

When two (or more) musical timbres are sounded at the same time, at any random interval between their perceived pitches, there is a high probability that at least some of the various partials from each will overlap. Figure 4 depicts a "Fusion Chart," as two such partials approach, begin to beat with one another, merge into one, and then repeat the sequence in reverse order as they separate and continue to move apart. A graphical representation of the related *critical bandwidth function* has been described by Pierce (1983, pp. 74–81) and Winckel (1967, pp. 134–148), and has its origins in Helmholtz's outstanding work (1954, chapter VIII) of a century ago. Helmholtz produced a chart of *harmoniousness of consonances* (1967, p. 193) that also forms the basis for Partch's similar representation which he whimsically termed *the one-footed bride* (1979, p. 155), and a simpler plot of *consonance* by Pierce (1983, p. 79). Our representation here takes on a bell-shaped curve which "objectifies" the normally subjective quality of dissonance into the more quantifiable phenomenon of beats and their frequency. (In certain unusual conditions that will not concern us here this simplification may need qualification.)

Surprisingly, our ears associate sound oscillations having a frequency of more than approximately 30

Fig. 4. Superimposed interval: Just major third (harmonic partials).

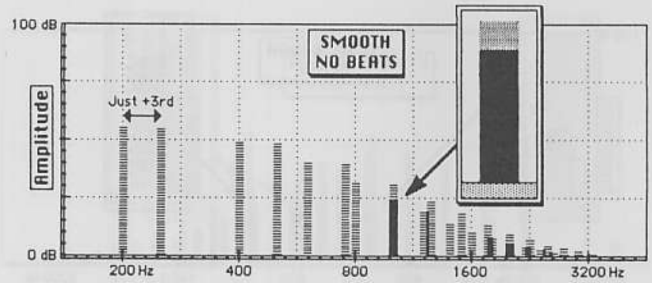
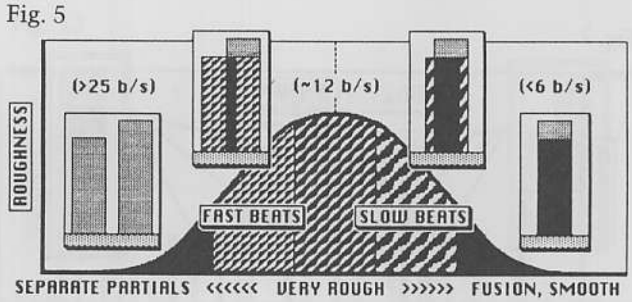


Fig. 5. Fusion chart: proximity of intervals versus roughness, with examples of enlarged bar graphs.

Fig. 5



Hz with musical *pitch*, while those somewhat below are heard instead as *loudness fluctuations*. Oscillations of about 0–6 Hz are usually described as *chorusing* or *tremolo*, and those around 6–16 as *roughness*. The remaining range, 16–30 Hz, can be quite subsonic and barely audible if soft, or at loud levels perceived as a sort of "unpitched rumbling," unless several (higher frequency) harmonics are present to give a genuine sense of pitch. Figure 4 indicates the straightforward graphic conventions we will be using for the rest of this paper.

Let's now play two simultaneous tones, both having the spectrum of Fig. 1 and tuned a Just major third (of 386.313714 . . . cents) apart. The resulting combination spectrum looks like Fig. 4. Both tones are shown with a grey cross-hatching so that where there is any overlapping it will appear as black, with beating shown as in Fig. 5, by graphically descriptive alternating bands or stripes. In this case the Just major third on harmonic partials is smooth, with no audible beating.

If the interval is now retuned to an equal-tempered major third (of 400.00 cents) as in Fig. 6, there is only a modest change in the relative locations of the lower partials. But between the fifth harmonic (here the term "harmonic" is accurate) of the lower tone and the fourth harmonic of the

Fig. 6. Superimposed interval: equal-tempered major third (harmonic partials).

Fig. 7. Superimposed interval: pure octave (harmonic partials).

Fig. 8. Superimposed interval: pure octave (stretched harmonic partials).

Fig. 9. Musical examples. (a) Very wide octave (1220 cents). (b) Very wide octaves (1212 cents each). (c) Chord and cluster (equal temperament versus Just). (d) The harmonic scale on C.

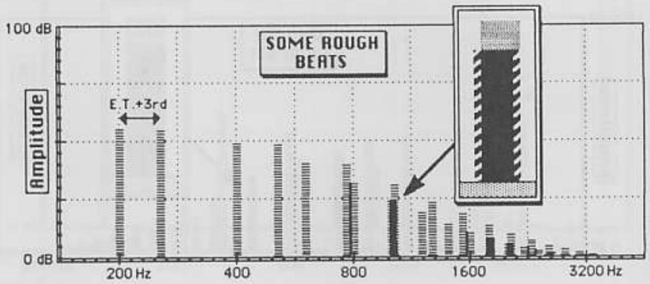


Fig. 7

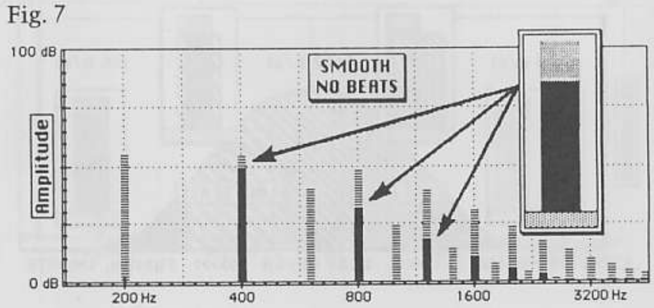
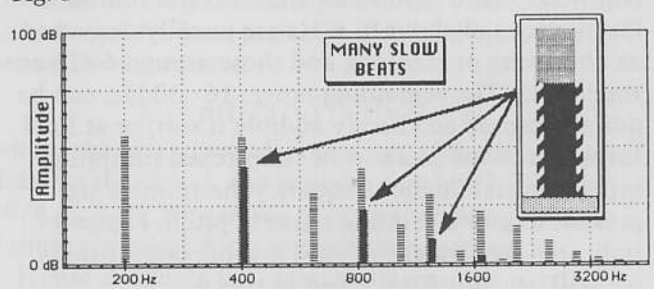
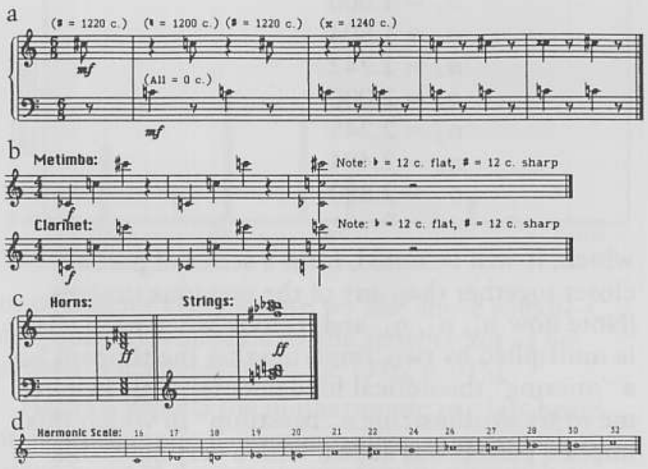


Fig. 8





upper one, some rough beating of 8 Hz occurs, this for the 200-Hz range we are using. Just a little higher the rate increases into the roughest area around 12 Hz, and above that the rate can even climb up into the audio band.

Now let's try a pure octave (of 1200.00 cents) and the same harmonic wave, as in Fig. 7. The result has, not unexpectedly, even more distinct "fusion" than the Just major third, with all harmonics lined up and no beats.

If we retain the same interval of an octave, but now substitute the stretched harmonic spectrum of Fig. 2, the result is as Fig. 8. Notice that there now appear many fairly strong beats between partials that had been "fused together" in Fig. 6. This is exactly the situation that exists with a piano, and leads directly to the stretched octaves mentioned earlier. There are many instruments that have even greater stretching of their partials than the piano, and the effect on octaves is more audible. One such instrument, which I call the "metimba," is a metal version of the (wooden) marimba, and like the "basstimp" alluded to earlier, arose in my timbre-synthesizing work from one of those usually impertinent "what if?" questions. Sound Example 2 on the soundsheet compares the metimba with a clarinet performing the same figuration (Fig. 9b): stretched-pure-stretched on the (stretched-harmonic) metimba, the last being a chord, and then pure-stretched-pure on the (harmonic) clarinet, again the last being a chord. Since the metimba has also the added complication of several nonharmonic partials (which we'll look at next), the smoothness of the somewhat wide octave is slightly disguised by them, but, then, this is a much wider octave than is found on a piano. In any event, the metimba is better served by a 6–12-cent stretched octave as in Fig. 10, while the clarinet sounds smoother with pure octaves, as in Fig. 7.

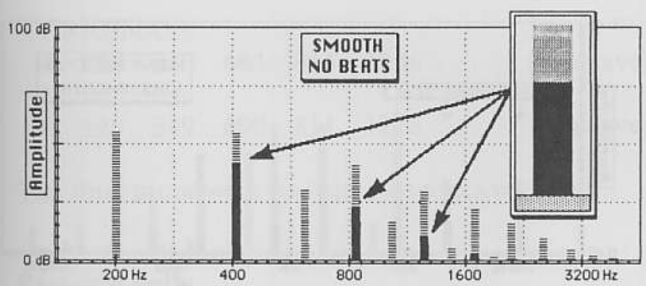
The more extreme case of *nonharmonic partials* is shown in Figs. 11 and 12, and heard as Sound Example 1 (see the soundsheet with this issue). Figure 11 plots the result of one particular nonharmonic timbre, that of Fig. 3, while Sound Example 1 compares a more typical nonharmonic timbre, gam (a replica of the tuned-kettles in a gamelan orchestra),

Fig. 10. Superimposed interval: stretched octave (stretched harmonic partials).

Fig. 11. Superimposed interval: pure octave (nonharmonic partials).

Fig. 12. Superimposed interval: very wide octave (nonharmonic partials).

Fig. 13. Superimposed interval: very wide octave (harmonic partials).



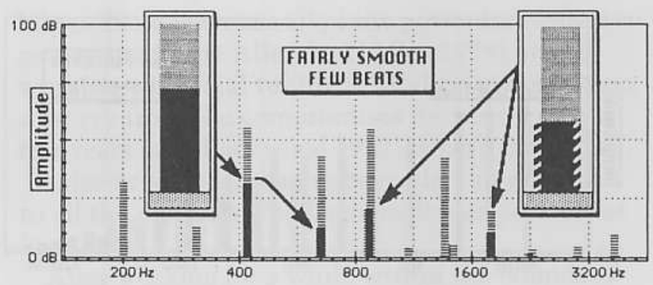


Fig. 11

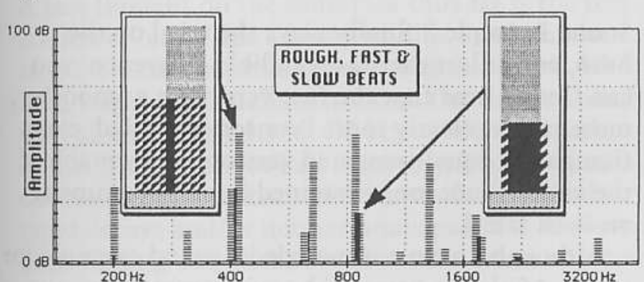
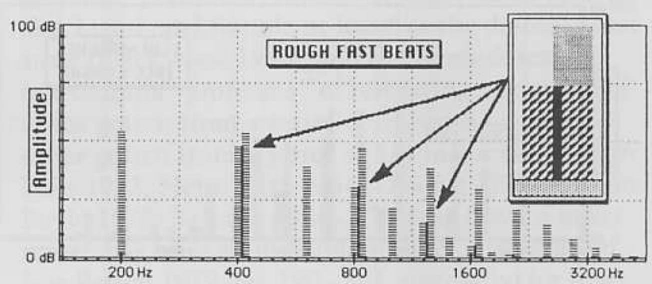


Fig. 13



with the same passage played on the French horn, and notated in Fig. 9a. As Fig. 11 indicates, a pure octave sounds fairly "rough" on most such instruments. We can imagine a nonharmonic timbre that would instead work better the other way around, with smaller-than-pure, <[2 : 1], octave, but most historically derived instrumental timbres have "accidently" been like the present example.

On Sound Example 1 you will hear that the lowest pitch, C4, is the reference lower note of the octave, and the upper three notes, all nominally C5 (they are closer together in pitch than the sharps and double-sharps in Fig. 9a visually suggest), are the tones of:

1. A 1220-cent "properly" wide octave
2. A pure octave of 1200 cents
3. The 1220-cent wide version again
4. A 1240-cent extremely wide octave

The differences are subtle, but gam does sound smoother on the 1220-cent wide version (like Fig. 12), while the horn is best at the pure 2 : 1 ratio octave. The horn has very audible beats when heard on the very wide octave, as in Fig. 13.

Clearly the timbre of an instrument strongly affects what tuning and scale sound best on that instrument, and exactly vice versa. I say scale because even though the graphs are of two simultaneous notes (and melodic intervals are much less critical), the soundsheet examples are first played melodically, and some of the same properties are represented musically here as well, if only for the octave.

Pierce (1983, pp. 192–193) discusses a corresponding example conceived of from a very different direction. He includes a recorded example (Ex. 4.4, and also listen to 2.1–2.5, and see p. 86), which demonstrates the aural illusion of a stretched-harmonic tone that sounds flatter when all of its partials are exactly doubled! If instead we were to move up by a properly stretched octave, as in our Sound Example 1, it would then sound like a "real" octave.

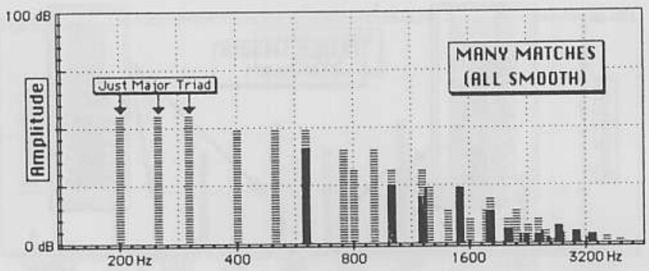
Triadic Confluence

Let's now add a third tone to the other two, to form a triad. In Fig. 14 we have the simplest case of a Just major triad played on a timbre that has harmonic partials. The result is smooth, as we would expect, because there are quite a few partials that merge and "fuse."

Fig. 14. Superimposed interval: Just triad (harmonic partials).

Fig. 15. Superimposed interval: Just triad (nonharmonic partials).

Fig. 16. Superimposed interval: nonharmonic "triad" (nonharmonic partials).



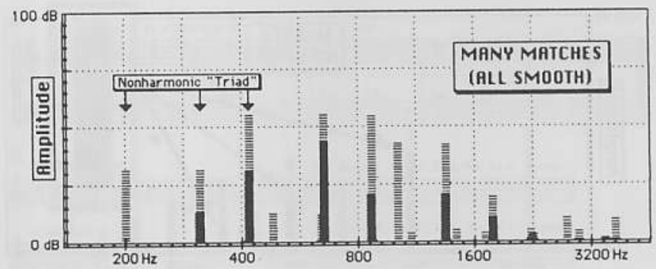
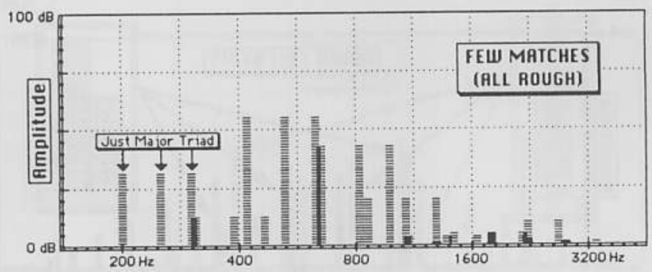


Fig. 15



If we try the same triad on three tones with the nonharmonic spectrum of Fig. 3, the result is not nearly so smooth. In fact, it's barely recognizable as the same fundamental harmonic building block of Western music! Figure 15 shows what happens, and you can listen to the effect on Sound Example 3. First we hear the triad with a traditional timbre ill-equipped to handle harmonies, odd as it might seem: the xylophone. You can judge the results for yourself, but while the xylophone has fewer conflicting partials than our Fig. 15's nonharmonic timbre, the "fusion" we normally expect to hear on a Just triad simply doesn't occur. You really can't even hear much difference, never mind choose one over the other, between the Just and equal-tempered versions!

By switching to another of my "whatifs," this time a metal version of the xylophone (metal-xylo), which has somewhat greater amplitudes of the higher partials, and all have a much longer decay-time than on a regular xylophone (metal soaks up higher frequencies more slowly than wood), the triad is slightly better defined. But the nonharmonic partials still allow us little preference between Just and equal-tempered versions, although some difference is at least audible. The last of Sound Example 3 finally plays the triad on the horn, and at last there's an audible difference: you can clearly hear that the Just version is a smoother, more stable, clearly more "consonant" triad, even though the equal-tempered version is acceptable if the chord is not long sustained (see the comment on Bach later).

Although I have not included a sound example for it, Fig. 16 shows a type of harmony worth investigating. Here we have a totally foreign and quite nonharmonic "triad," played with the same nonharmonic partials as in Fig. 15, where the Just triad sounds so diffuse. But the results here are surprising to an extreme: the peculiar "triad" is consonant, fused and well focused, with very little roughness, and many matches of partials in all their nonharmonic glory.

This is the kind of harmony we find in the gamelan music of Bali and Java, where the scales are pelog and slendro, and the instruments are tuned gongs, kettles, and metalphones. On my latest recording, *Beauty In The Beast,* is the composition (in homage to that magical island) I call: *Poem for Bali.* In it these timbres and scales are combined with good effect, to produce a music that is neither Western nor Indonesian but is from both traditions. Near the end a *Concerto for Gamelan and (Symphony) Orchestra* occurs. In order to get the two groups (both actually synthesized "replicas," of course) to be able to "play" together, it was necessary to "cheat" the pelog scale slightly toward Just intervals so that the (mostly) harmonic symphonic timbres would not sound cacophonous when playing it, and yet not so far that the gamelan ensemble would suffer. The original pelog and modified "harmonic" versions are (in cents):

ORIGINAL:
0 123 530 683 814 1226 (= octave)
MODIFIED:
0 123 519 690 811 1216 (= octave)

The final piece with its "cheat" works very well.

Partch's Folly

A last thought on the examples thus far is the realization of one of life's funny ironies. I only understood it recently while researching this paper. It's both tragic and touching to think that Harry Partch, who almost single-handedly kept alive the spark of Just intonation for half a century, happened to choose to build instruments most of which had rapid decays and/or nonharmonic partials. As we've been finding out, these are about the worst choices that can be made to best show off the wonders of Just tuning. J. S. Bach demonstrated, and Benade alludes to it in the quote earlier on, when you must use slightly defective tunings, like $12\sqrt{2}$, it sounds a lot less rough if you keep everything in motion: don't linger on the imperfections. . . . Tempo is very much tied up with tuning. Both evolve together and given a better tuning than $12\sqrt{2}$, we might expect slower or at least more sustained tempi to evolve than we now use. But Partch tended to compose music that generally moved right along, as we must do in much equal-tempered music—which only further "hid" the beauty of his commendable 43-note-per-octave scale. Life plays tricks on us all.

The Harmonic Scale

Sound Example 4 of the soundsheet allows us to compare a chord played alternately on $12\sqrt{2}$ equal temperament, and on a sort of "super-Just" tuning I've been using called *the harmonic scale.* First I had to write a series of programs which access the tuning tables of the Synergy-Plus and GDS Digital Synthesizers (Kaplan 1981) which I use, once Stoney Stockell (who developed the final software and much of the hardware for both these systems) found a way to open these up to my curious bumblings. Now I can use all of the power for timbre generation of the Alles card (Alles 1979), while working with total tuning flexibility and precision, a far cry from the compromises necessary only a few years ago (Blackwood 1982), where the choice, "scales-or-sounds," had to be made. I'm grateful to all the pioneers who have made this possible at long last.

After working for a while within the bounds of traditional Just intonation, I became frustrated with both the "plainness" (to ears brought up on Stravinsky, Ligeti, and Bartok, at least) of the diatonic Just scale (Blackwood 1985), and the often-described one-comma "problems" of relegating one group of notes gotten from a series of 3/2 fifths with another gotten from a series of 5/4 major thirds (Barbour 1953; Helmholtz 1954; Olson 1967; Lloyd and Boyle 1979; Balzano 1980). That the "twain never meet" has been an indelible source of frustration (see Partch 1979, pp. 190–194 especially) for Just-tempted musician and theoretician alike.

But if we are willing to give up the ability to modulate, and solve that with the computer hardware, we can tune all 12 steps of our Synergy keyboards or whatever to a series of pitches from the overtone series of one particular tonic, as I have done with the harmonic scale (see Fig. 9[d]). Any selection of pitches, and indeed all of these pitches when sounded together fuse as consonantly as does the triad of Fig. 14 (at least for harmonically partialed timbres). The particular choice of 12 pitches in Fig. 9(d) maintains the greatest continuity of successive prime partials, up to number 19, in fact. (Note that number 21 is a 7/4 above number 12, G an octave lower, and that number 27 is a 3/2 above number 18, i.e., they are not musically "prime.") These are shown in Table 1.

Figure 9(c) shows the notes played on Sound Example 4 and 5, both of which alternate $12\sqrt{2}$ and harmonic-scale tuning, and give you a good comparison of the two. There's little doubt that the equal-tempered versions are inferior. You can hear especially well in the high string-cluster example the very low difference-tone fundamental, a natural effect of everything being all in tune at last.

And by calculating the same sequence of harmonics for each of the 12 basic starting pitches, trans-

Table 1. Harmonic scale on C table

Note	*Ratio*	*Cents*
C♮	1/1	0.000
D♭	17/16	104.955
D♮	9/8	203.910
D♯E♭	19/16	297.513
E♮	5/4	386.314
F♮	21/16	470.781
F♯	11/8	551.318
G♮	3/2	701.955
A♭	13/8	840.528
A♮	27/16	905.865
B♭	7/4	968.826
B♮	15/8	1088.269

posing it up and down by 100-cent increments, we arrive at 144 distinct pitches to the octave for our new "modulating" version of the harmonic scale. These are loaded into the Synergy frequency tables (in groups of 12) via an outboard Hewlett-Packard computer and a single-octave keyboard which I've built (Milano 1986) to trigger the computer each time I key a change to a new chord fundamental. The result (which had classically been called impossible) is that we can now modulate completely around a "circle of fifths" while at all times retaining not only the few "classic" Just intervals, but all the other perfectly tuned pitches of the harmonic scale. And that's precisely what happens twice in my recent composition, *Just Imaginings* (on *Beauty In The Beast,* Aud 200). An excerpt of the first "circle" is heard in Sound Example 6.

The Best and Worst

Earlier I suggested about meantone that we have excellent reasons to believe that were it not for modulation problems of "wolf tones" (when instruments were tuned to only 12 meantone pitches), this form of temperament would likely be in use right through to the present. In Sound Example 7 you can get some small sense of the frustrations that must have occurred for musicians of J. S. Bach's day as the inevitable "pollution" of their triads first occurred, in deference to the instrument manufacturers who drew the line at 12 notes per octave. You can hear the actual derivation of the still-in-use approximation: two octaves down and four (slightly flat) perfect fifths up (or in the opposite order) ≈ a major third, first as it sounded in meantone, and then as it sounded in the then-new equal temperament. There's little doubt that ears accustomed to the smoothness of the former would have detected the falsity in the latter more acutely than we can today, although the difference is plain enough if you listen attentively.

What is actually occurring is that we are trying to force $(3/2)^4(1/2)^2$ to be = 5/4. The left terms total 81/64, or 81/80 higher than 5/4. In cents, the left side is $(4 * 701.9950) - (2 * 1200.0) = 407.8200$, and that is greater than 386.3137 by the *syntonic comma* of 21.5063. But musicians for centuries have notated the major triad on C to be: C–E–G, not: C–E-flattened-by-one-comma–G, that is: C–E_{-1}–G (Barbour 1953, chapter III; Blackwood 1985, pp. 154–162; Helmholtz 1954, pp. 430–439). Something had to give, and the best-sounding way was meantone. There are four fifths to absorb the excess comma, so let's give each a quarter of it: $701.9950 - (21.5063/4) = 696.5784$ for the meantone fifth. (Two of these up [and down an octave] form a "tone" which is exactly in between the "major tone" of 9/8 and the "minor tone" of 10/9, hence the name, *mean*tone. . . .) Since low partials are involved, the resulting flat fifths beat much slower than if some higher ratio were mistuned by the same 5.3766 cents—if you will, the fifth can "stand it" better. The major third that results from four of these fifths is, as you can hear on the first half of Sound Example 7, perfect: 386.3137 cents.

To round out this capsule explanation, what occurs in the equal-tempered version is easily shown. By definition the size of an equal-tempered fifth is: $2^{(7/12)} = 1.49830708$, which is certainly close to 3/2 (and why we can get away with it as the only deviation, always melodically, in the 144 notes per octave modulating harmonic scale). In cents we have the familiar fifth, 700.0. Now, go up four of these and down two octaves: $(4 * 700.0) - (2 * 1200.0) = 400.0$. And this is $400.0000 - 386.3137 = 13.6863$ cents sharp for a major third. A 5/4 involves harmonics double those for a 3/2, so the beats of this sharp third are worse than only four times the meantone fifth's error, as you can hear on the second part of Sound Example 7.

By the way, to get a feel for the ratio form of describing Just intervals, think of them as naming which two partials of the overtone series form the interval—for example, if it were the seventh and the fourth (like B-flat$_{7\text{-har}}$ and C) the ratio would be 7/4.

The next brief Sound Example 8 places the smooth quality of meantone into a more musical context. By employing the same suggested technique as the harmonic scale above, retuning as needed via an external computer control, we can maintain this tuning's benefits to most of the existing repertoire, while avoiding both "wolf tones" and any restrictions on modulation. I expect to see such an evolutionary version of meantone become more common as digital synthesizing equipment of this kind becomes readily available.

But for traditional Western music the worst possible way to tune is probably the scale of 13 equal steps in an octave ($13\sqrt{2}$), which is heard in the Sound Example 9 (with the same music as the last to aid in comparisons). As we will next see in equal divisions of the octave, right beside each decent one is a region of pretty awful ones. And the lower the number of steps, in general the less good the results. Since $12\sqrt{2}$ is the lowest decent division, right beside it somewhere there ought to be a "dilly," and this one's it! (Yes, a new tool's strength works both ways. . . .)

Symmetric Equal Divisions

This could eventually be one of the most fruitful methods of scale development for our new musical technologies. The notion of dividing the octave into steps of equal size also happens to have gotten a great deal of attention over the past few hundred years, so there's really a good deal of information already available about it (Barbour 1953; Helmholtz 1954; Yasser 1975; Partch 1979; Yunik and Swift 1980; Blackwood 1982, 1985). As an idea, multiple division seems to have arisen independently with several musical theorists, as early as the sixteenth century in Italy. (Zarlino and Salinas both suggested a division of 19 parts, although it's doubtful that their derivations were exactly equal [Barbour 1953, p. 115].)

In 1555 Nicola Vicentino described what we now recognize as the 31-note division. It was rediscovered mathematically in Holland by Christian Huygens in 1724, who gave an exact description of his "harmonic cycle" and how extremely close it comes to standard quarter-comma meantone intonation. As such it has all the merits of meantone for performing most Western music far more smoothly than $12\sqrt{2}$, while also permitting unlimited modulations to an unprecedented degree (to the nearest 1200/31 = 38.7097 cents!).

But the distinction of "oldest" genuinely equal-stepped division probably goes to the excellent 53-note division. A student of Pythagorus named Philolaus wrote a description of a method for constructing intervals which leads directly to a *measuring scale* of 53 comma-sized steps, in which form it is known as *Mercator's cycle* (Helmholtz 1954, p. 436). Just over a century ago this division was championed by R. H. M. Bosanquet, who developed the ideal keyboard with which all possible regular divisions become really playable by human hands, the generalized keyboard (Helmholtz 1954, pp. 479–481; Yunik and Swift 1980, Fig. 1). Even Partch (1979, pp. 393 and 438), who had little love for any equal division, has kind words for Bosanquet's generalized keyboard.

There is scarcely a more worthwhile venture to pursue as soon as possible than adopting a standard for and then manufacturing at least a "limited edition" of these keyboards for all of us now becoming involved with this field. The present author constructed a generalized keyboard (which has subsequently been lost) back in the presynthesizer days of 1957 and would love to hear of any serious work now being done in this direction. I have several long-thought-out ideas and proposals (see Fig. 17), which I will gladly share towards the goal of "getting the standard right," that we may avoid the fate of digital tape recording. Obviously it ought be some variant of a MIDI general-purpose unit.

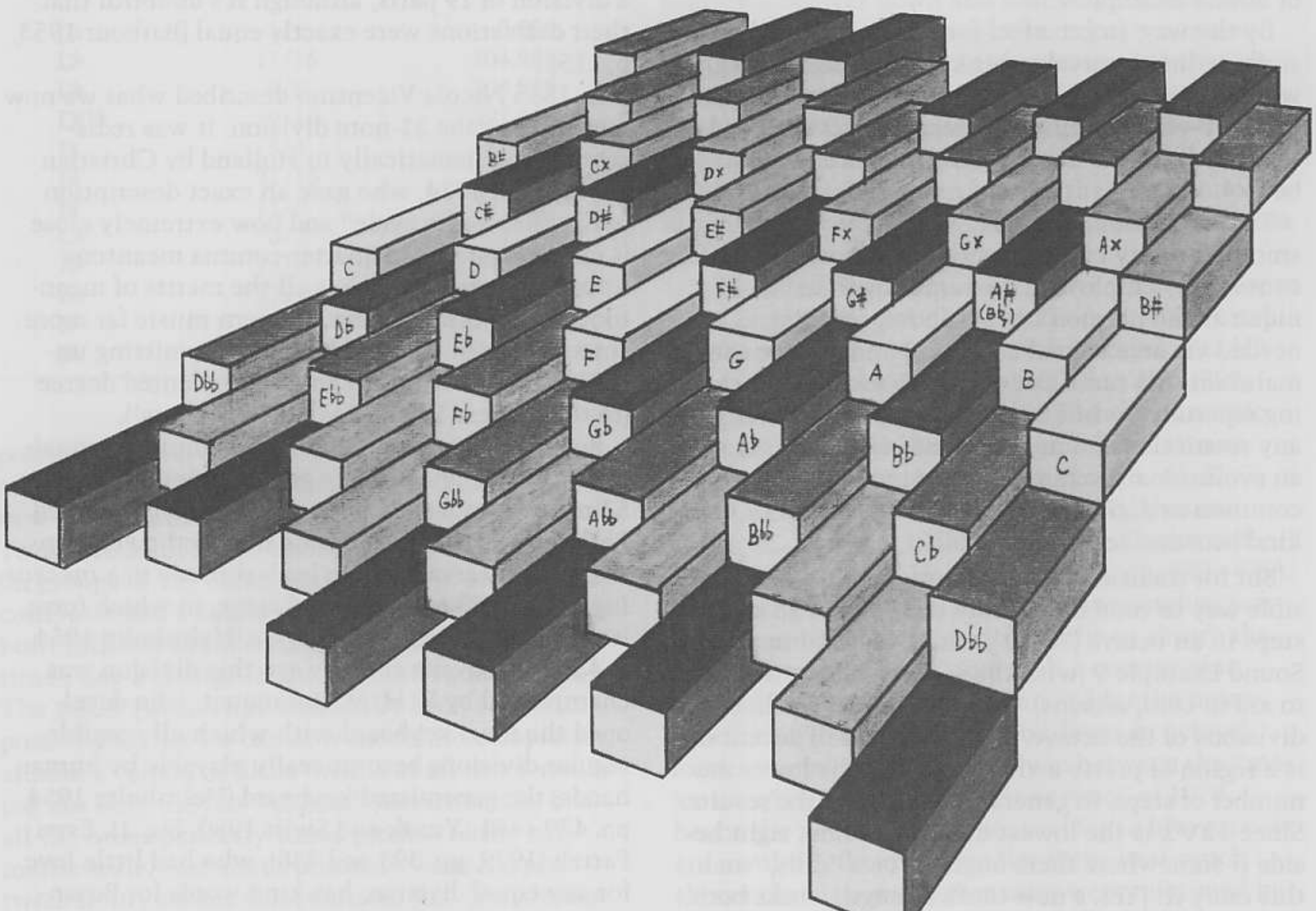


Fig. 17. "Multiphonic" generalized keyboard. One octave, C to circa $\sqrt[31]{2}$ note names. This is a perspective drawing of one octave from a proposed standard of generalized keyboard by Wendy Carlos. The individual notes are shaped like the sharps and flats of a standard keyboard, approx. 1/2" wide and 1/2" high, but only 1-3/4" long. They are colored in five shades along the blue-yellow axis of the CIE Chromaticity Diagram, to facilitate proper use by the color-blind. The suggested shades are: white (for naturals), black (for sharps), dark brown (for flats), light greyish-blue (for double-sharps), and golden tan (for double flats). The notes are named here for the $\sqrt[31]{2}$ Division, but the keyboard is suitable for all regular divisions of the octave, up to 55 notes, the number of digits per octave here, although it may be extended as necessary. This design is a modern updated treatment of the original, conceived and built in 1875 by R. H. M. Bosanquet.

Which equal divisions look most useful? Figure 18 is a computer plot I recently made to help narrow down the decision. The inspiration came from a simpler version by Yunik and Swift (1980, p. 63) in *Computer Music Journal* 4(4). But I was curious about what happened as one went continually and gradually from integer step to integer step, and also wanted to reflect the fact that with a computerized frequency-table driver, there's little need to "penalize" what may be a fine division that requires more than two dozen notes (although more than 60 could get unwieldy). The computer certainly doesn't care, and if the task can be somewhat automated by such, why should we care?

Since harmonies from the natural seventh of 7/4 (septimal), and perhaps from the natural eleventh of 11/8 (unidecimal) may play a good role in any newly developing harmonies, I've added these onto the more common "classic-Just" ratios as two options to be additionally calculated and plotted. My algorithm ended up being quite different from Yunik and Swift's:

1. Calculate and store the cents for each prime ratio of interest. (I chose: 3/2, 4/3, 5/4, 6/5,

Fig. 18. Equal divisions of the octave. Σ(deviations from pure)².

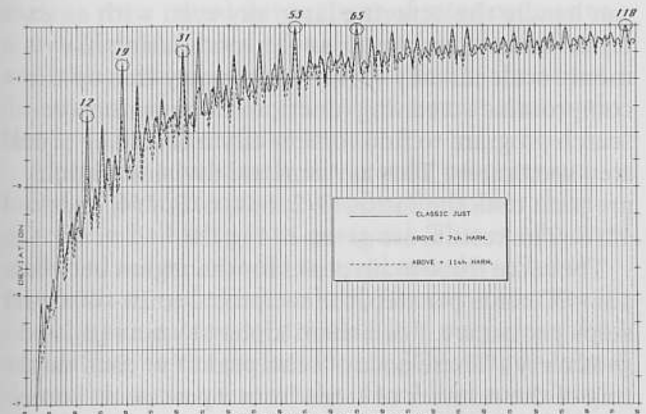
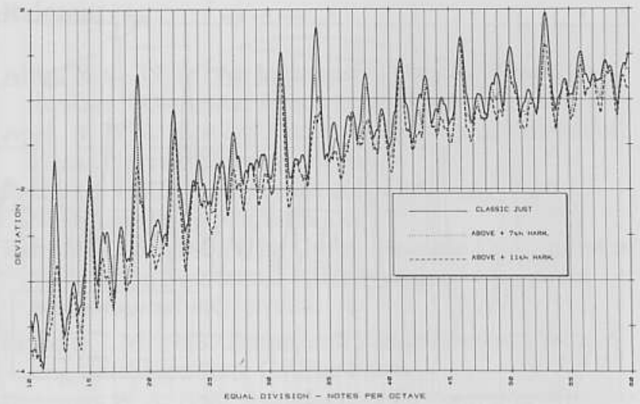


Fig. 19. Symmetric equal-step divisions from 10 to 60 notes per octave.



5/3, 8/5, plus 7/4 and 11/8. The rest can be derived from these and are not prime.)

2. Set up a loop with some small increment on division size.
3. For each new division size, calculate the size of one step, and set up and clear the proper registers.
4. Find the nearest step to each given ratio and calculate its error.
5. Add the square of this error to the "sum-of-the-squares" register.
6. Plot that value, increment the division size, and loop to 3.

The plot is carried out three times: once for the "classic-Just" ratios, once with the additional restrictions of satisfying the septimal ratio (7/4), and a third time adding also the unidecimal ratio (11/8).

The resulting Fig. 18 fits very well my own and all other examined descriptions of the audible harmonic "merit" of particular equal divisions. Note the peaks singled out at several of the most important divisions: 12, 19, 31, 53, 65, and the one that is a mean between 53 and 65, 118 (= 53 + 65, and is "nearly perfect" by all published reports and by our diagram). Also notice that we are not searching in particular for diatonic scales (Blackwood 1985), another reason for omitting 9/8 and 10/9 from the list in step 1 of the previously mentioned algorithm. A much closer view over the range of 10 through 60 stepped divisions is given in Fig. 19. I think we will want initially to experience those divisions with less than 60 steps before heading for the really elaborate alternatives.

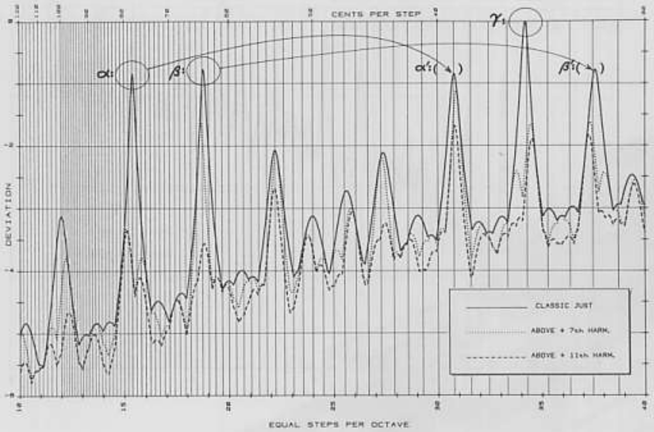
The solid line is of the "classic-Just" deviations only. The dotted line is that for septimal harmonies added, and it obviously never goes above the former, although at times it can get quite close, as at the otherwise so-so 15 division, or at the remarkable 31 harmonic cycle of Huygens. Similarly the dashed line adds the harmonies from the 11th partial, and again fits well at those same two divisions, 15 and 31.

The most thorough investigation of cycles 13–24 was likely that recently done by Blackwood (1982 and 1985). He also took the extra step and composed a series of etudes, one in each cycle. It's very important that we do not fall into the trap (Lloyd and Boyle) of remaining only theorists. We have to compose real music of many kinds within all and any of our new tuning schemes, if this work is to have any lasting value at all, or be taken seriously by the music community (and by the public at large, if it is to survive). More importantly, just as "Papa Bach" did with his generation's new scale, this is the only way we're going to learn how to control and use wisely these new gifts from the age of the computer to us!

Asymmetric Divisions

I've saved the most surprising tuning concept for last. The previous divisions all made the same tacit

Fig. 20. *Asymmetric equal-step divisions, locations of alpha, beta, and gamma.*



assumption: that each target ratio be available as an interval in all inversions. They were symmetric: there was the prime ratio of the perfect fifth, 3/2, but also the perfect fourth, 4/3 (given 2/1 and 3/2 it follows directly, so is not really "prime" in the sense we've been using that idea). Similarly, there was the major third, 5/4, but also its inversion, the minor sixth, 8/5. Both 6/5 and 5/3 appeared. Only for the newer optional ratios is a nonredundant form applied: we have 7/4 but not 8/7, 11/8 but not 16/11.

Since each of the redundant pairs is symmetric with respect to the octave, the net result is sort of an "over-representation" of this interval. Little wonder that every one of the peaks to the plots in Fig. 18 and 19 occurs at exact integer divisions of the octave! But the octave is the ratio most common to the "strategies" of most digital synthesizer architectures, such as in the 16′, 8′, 4′ octaving borrowed from the pipe organ. Most timbre/instrument files include the similar designation of transpositions up or down by octaves. In my current Synergies the frequency of each note is stored in the previously mentioned frequency-table, but only 12 main 16-bit words control the middle-most octave, the other octaves duplicating these frequencies by exact factors of two, a very common method of assuring all the octaves can be made "beat-free." We have octave possibilities all over the place.

So why not, as an experiment, omit the octave-redundant ratios from the first step of our algorithm? That will lose all octave symmetry, but if we can handle the octaving later, say even with an external control computer as the modulating harmonic scale already requires, that may free up the compromise-screening functions of our intensive-search program to find some really interesting equal-step specimens. This asymmetric division search program uses the ratios: 3/2, 5/4, 6/5, 7/4, and 11/8. The results are given in Fig. 20.

There are three main peaks for the region between 120–30 cents per step, the reciprocal to 10–40 equal steps per octave. I call them alpha (α), beta (β), and gamma (γ). There occur "echo peaks" at each doubling of the number of steps from a past peak, something difficult to see in the symmetric division plots. The two in Fig. 20, α' and β', fall equally to either side of the essentially perfect(!) γ (on the classic-Just curve—for septimal harmonies the first "echo," α' is excellent). These happy discoveries occur at: α = 78.0 cents/step = 15.385 steps/octave, β = 63.8 cents/step = 18.809 steps/octave, γ = 35.1 cents/step = 34.188 steps/octave. The deviation axis's arbitrary units are incompatible on this figure with those on the last two, but a quick comparison will show just how different (and also similar) the asymmetric and symmetric divisions are.

Sound Example 10 plays a "nearly" one-octave scale of alpha on the horn. Notice how there are four steps to the minor third, five steps to the major third, and nine steps to the (this time no kidding) perfect fifth, but, of course, no octave (the final "attempt" at this is an awful 1170-cent version, the next step to 1248 cents being even further away!). But that's the trade-off we've requested: there's no free lunch! Sound Example 11 is a brief chordal passage in alpha. The harmonies are amazingly pure; the melodic motion amazingly exotic.

I've not included a similar example of beta or gamma on the soundsheet but have experimented with both (gamma really requires a "multiphonic" generalized keyboard, like most <24 divisions). Beta is very like alpha in its harmonies, but with five steps to the minor third, six to the major third, and eleven to the perfect fifth, melodic motions are different, rather more diatonic in effect than alpha. Gamma (nine steps, eleven steps, twenty steps) is slightly smoother than these, having no palpable difference from Just tuning in harmonies. But the scale is yet a "third flavor," sort of intermediate to α and β, although a diatonic scale is melodically available. I have searched but can find no previous description of α, β, or γ nor their asymmetric scale family in any of the literature.

Alpha has a musically interesting property not found in Western music: it splits the minor third exactly in half (also into quarters). This is what initially led me to look for it, and I merely called it my "split minor-third scale of 78-cent steps. Beta, like the symmetric 19 division it is near, does the same thing to the perfect fourth. This whole formal discovery came a few weeks after I had completed the album, *Beauty In The Beast,* which is wholly in new tunings and timbres. The title cut from the album contains an extended study of some β, but mostly α. An excerpt highlighting the properties we've been discussing is the final Sound Example 12.

Conclusion

I don't pretend to be either a writer or theorist, so thank you for putting up with my attempts at both in this lengthy report. But I have been fascinated by scales and tuning for about thirty years, well before discovering electronic music, with the single exception of Olson's (1967, chapter 10) demonstration record of the RCA Synthesizer in 1953. Although my name is indelibly connected with the Moog Synthesizer since that unexpected accident of some 18 years ago, it seems to me that only now is our fledgling art starting to show healthy signs of growing up. I'm passionately excited by the promise of: any possible timbre, any possible tuning. That's the reason for the work that went into this paper. It's also a way to return the favor to *Computer Music Journal* for a decade of ideas and inspiration that led directly to my "very nearly" attaining this double-headed ability. For all those who helped and those on whose shoulders I am lucky to stand, deepest thanks. We've been treading water much too long, and I'm delighted that the real work now can begin.

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